Designed to Fail: The Medicare Auction for Durable Medical Equipment

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Abstract

We examine the theoretical properties of the auction for Medicare Durable Medical Equipment. Two unusual features of the Medicare auction are 1) bids are non-binding and 2) winners are paid the median winning bid. These two features lead to complete market failure. Lowball bids result in a price that is below each bidder's cost, so no quantity is supplied. In sharp contrast, the standard clearing-price auction has each bidder bid true costs as a dominant strategy, resulting in competitive equilibrium prices and full efficiency. Recent Caltech experiments (Merlob et al. 2011) confirm these theoretical findings.

Introduction

The Centers for Medicare and Medicaid Services (CMS) conducted auctions in nine major metropolitan areas in November 2009 to establish prices and identify suppliers for durable medical equipment. The impetus for these auctions was the 1997 Balanced Budget Act, which specified that competitive bidding be used as a means of "harnessing market forces" to decrease Medicare costs. The prices that resulted from the 2009 auctions took effect on 1 January 2011 and are substantially lower than the legislatively set reimbursement prices used previously. Unfortunately, as we show here, these price decreases are artifacts of a flawed bidding system that encourages lowball bids that do not reflect suppliers' costs. The price reductions likely come at the expense of diminished quality and service to Medicare beneficiaries. These apparent price reductions are short-sighted since supply shortages, poor quality, and increased fraud are likely to increase total cost as Medicare beneficiaries are forced into more expensive options.

The main problem with the CMS auction is that it did not undergo rigorous vetting by auction design experts during its formative stages. Such vetting has proved highly successful when, for

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example, designing auctions for electricity and radio spectrum.[2] Alternatively, auction experts were noticeably absent from the CMS process which is unfortunate since there is near consensus amongst such experts that the CMS design has fatal flaws that could have easily been avoided. These flaws are so substantial that 244 prominent experts (including four Nobel Laureates) cosigned a 17 June 2011 letter to President Obama urging an executive mandate to restructure the CMS auction.

In their letter the experts appealed to the President's Executive Order of 18 January 2011 that stressed regulation be transparent and based on the best available science, pointing out that the CMS program is the antithesis of both. The experts' opinion is based on intuition steeped in auction theory, experimental and empirical studies, and decades of experience in auction design and implementation. Unfortunately, mere expert advice has not convinced CMS to change course. Thus the purpose of our paper: to convey the flaws in the CMS auctions while maintaining a scientific rigor that CMS cannot dismiss.

To better understand the intuition behind our results, it is important to understand the evolution of the CMS auctions. CMS began auction pilots as a means of setting reimbursement prices for Durable Medical Equipment in 1999 in Demonstration Projects in Florida and Texas. In those pilots, and still today, firms bid to supply a variety of products within specified categories and the winning bids on each product are used to set the price at which Medicare providers are reimbursed (Medicare does not reimburse non-winning bidders). Bidders compete based on a "composite" bid that is a weighted average of their bids on the different products in a category where the weights indicate the relative importance of the product to the category.[3] CMS selects winners beginning with the lowest composite bid and works upward until the total capacity of winners is sufficient to satisfy estimated demand in the category.[4] However, winning the auction does not mean that a bidder becomes a Medicare supplier. Rather, winning the auction simply earns a bidder the option of signing a supply contract—which the winner is free to decline since bids are not binding.

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[2] See Ausubel and Cramton (2010) and MacMillan (1994) respectively.

[3] Katzman and McGeary (2008) show that the composite bid rule leads to inefficiencies as it provides strong incentives for bidders to skew bids away from costs.

[4] In response to lowball bids (which are predicted by our model), CMS manipulated the number of suppliers it felt necessary to fill demand in order to work its way up the supply schedule and produce a more "realistic" price—a price that nonetheless is arbitrary and no better a reflection of competition than the previously set administrative prices.

Over the past eleven years, CMS has altered the bidding rules in a variety of ways to address apparent problems. For example, reimbursement prices were initially set equal to an upwardly adjusted average of the winners' bids on individual products. When wide price swings were observed in the first demonstration project, CMS imposed ceilings and floors[5] on allowable bids and changed the weighting system it used for calculating composite bids. Later, CMS became disenchanted with the average pricing rule and switched to setting the price equal to the median winning bid in the 2009 pilots and plans to use median pricing as the program expands nationwide.

Several economists have expressed to us that despite the seemingly obvious problems with its auction, CMS must have good reasons for the auction rules. Unfortunately, that simply is not the case. Rather, the CMS decision process has been riddled with poor economic reasoning. Specifically, when comparing the median price to the clearing price rule, CMS states "the median pricing rule *would not have affected* the number of winning bidders who signed contracts or *the suppliers' bidding strategies*" (Federal Register, 2007, page 18078).

The problem with the CMS assumption is that it is wrong. Bidders do indeed bid differently when faced with different pricing rules. For this and other reasons Cramton and Katzman (2010) argue that CMS must fundamentally change to an auction that respects basic principles of auction design, not just further adjust a poorly designed system that is based on bad economic logic. Cramton (2011) suggests a dynamic clearing price auction as a preferable alternative that can be easily implemented.

Within our setting, Cramton's (2011) dynamic auction is isomorphic to Vickrey's (1962) sealed bid clearing price auction where the price is selected to balance supply and demand.[6] Our analysis begins with a review of the clearing-price auction. When bids are binding, the clearing-price auction elicits the dominant strategy equilibrium where bids equal costs. The lowest-cost bidders provide the goods and full economic efficiency is achieved. However, we show that despite these appealing properties, if bids are not binding, then many new (un-dominated) lowball bidding equilibria emerge, making it difficult to predict outcomes and calling the efficiency of this renowned auction into question.

[5] The bid ceiling is a reserve price, which is standard in auctions, but the fact that CMS needed to set bid floors to keep bidders from bidding too low should have been a clear sign that something was wrong with the CMS auction format.

[6] The dynamic clearing-price auction differs from Vickrey's auction in more complex environments.

As in the clearing-price auction, we find that if bids are not binding, then a multitude of lowball bid equilibria also emerge in the median-price auction. However, unlike the clearing-price auction, we show that even if bids are binding, then no equilibrium bid function exists that (1) is an increasing function of costs and (2) does not violate the CMS set bid ceiling. This results in inefficient outcomes. We therefore conclude that simply altering the current CMS process to make bids binding is not sufficient because bids at auction would still not represent costs, some low-cost bidders would still not win, and some providers may be forced to supply the product at a price below their cost.

Our claims are not just theoretical. The clearing-price auction (with binding bids) is widely used and studied. Its desirable properties are well understood both in the field and in the experimental laboratory.[7] Dynamic implementations of the clearing-price auction have performed especially well in the field (Ausubel and Cramton 2004) and in the lab (Cramton et al. 2010). However, until Merlob et al. (2011), no one had experimentally examined the impact that non-binding bids have on the clearing-price auction since auctions with non-binding bids are so rare in practice.[8] Interestingly, they show that non-binding bids encourage lowball bids even in the usually efficient clearing-price auctions. This is consistent with our finding equilibria with lowball bids in that auction. Thus, the theory and experiments agree and elucidate the poor incentives created when the auctioneer ignores one of the most basic principles of auction design: bids must be binding commitments.

Merlob et al. (2011) also shed light on bidder behavior in the median-price auction which has never been studied before since no one other than CMS has ever used such an auction format. Once again, the experimental results are consistent with our theoretical predictions. They found that bidders in median-price auctions with non-binding bids were likely to submit lowball bids; we find multiple lowball bidding equilibria in this case. The experiments also show that when bids are binding, a large number of bids in the median-price auction bump up against the bid ceiling. This is consistent with our theoretical predictions. The experiments thus reinforce our claim that using the median-price auction compounds the problems created by non-binding bids. Simply fixing one part of the CMS auction is insufficient.

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[7] For experimental results, see for example, Coppinger et al. (1980), Cox et al. (1982), Kagel (1995) and Kagel and Levin (1993, 2001, 2008).

[8] One notorious exception is the April 1993 auction for Australian satellite television services. Because bids were not binding these auctions were marred by bidder default and political embarrassment. See McMillan (1994) for details.

Using theory to better understand the properties of the CMS auction is important because of the auction's use in the field. The CMS auction represents an important test case of the application of market methods for the provision of certain Medicare supplies and services. Failure of the approach might well discourage the further application of market methods. Our hope is that this theoretical work will be part of the scientific evidence that establishes the need for reform of the CMS auctions. The problem is not that auctions do not work in Medicare, but that the CMS auction is badly flawed.

How did CMS come up with such a flawed design? One possibility is that CMS was thinking along the lines of a clearing-price auction, but given that a key motivation of the auction was to cut costs, decision makers felt uncomfortable with paying all the winners more than they bid. They viewed the median of the winning bids as a fairer market consensus of what the price should be. This would involve paying some winners less than their bids, so a fix would be to ask them if they would like to supply at the median price, and not penalize those who say no. A naïve administrator might be swayed by such logic. The flaw in the argument is a failure to recognize the incentives the approach creates for the bidder. The bidder does best by bidding the floor and thereby gaining a free option to supply at the median price. Such lowball bidding means that the bids do not reflect costs, but only the administratively determined bid floor.

In summary, our analysis highlights three main points. First, if bids are not binding a multitude of equilibria arise in both the median-price and clearing-price auctions, leading to a multitude of possible outcomes the totality of which is unpredictable. Since lowball bids bias prices downward, large inefficiencies can result as bidders simply decline the CMS contract because the price is too low.[9] Second, even if bids are binding, the CMS median-pricing rule and bid ceiling lead to non-existence of bounded, increasing equilibrium bid functions, which also leads to inefficient supply. The practical implication is that Medicare beneficiaries will have difficulties acquiring products and those products that can be obtained are apt to be of poor quality. Finally, we note that a clearing-price auction with binding bids generates efficient outcomes and eliminates the problem of shortage, thus providing an alternative format that is easily implemented.

[9] Two sources of inefficiency are present. The first is that too little quantity will be supplied. Second, even if the efficient quantity is supplied, it is unlikely that the lowest-cost bidders will win.

The model

We consider an Independent Private Values (IPV) model where only one product is to be supplied and multiple winners are chosen. N risk-neutral bidders have unit capacities and an odd number, $W(<N)$, of suppliers is necessary to fulfill demand.[10] Bidder i's cost of providing a unit of the product is $c_i \in [L,H]$ which is drawn from the cumulative distribution function $F(c)$ with corresponding density $f(c)>0$. It is assumed that $f(c)$ has derivatives of all orders on $[L,H]$ and we denote the minimum and maximum values that $f(c)$ obtains on $[L,H]$ by $f_{\min}$ and $f_{\max}$ respectively.

The reimbursement price on units supplied equals the median winning bid under the median-pricing rule and equals the $W+1^{\text{st}}$ lowest bid in the clearing-price auction. Bids are restricted to be no lower than $\underline{b}\,(\le L)$ and no higher than $\bar{b}\,(\ge H)$. At times it will be useful to order the costs of a bidder's rivals $c_{(1)} < c_{(2)} < \cdots < c_{(N-1)}$ so that we can use the properties of order statistics. To that end, let $F_k(x)$ denote the distribution of the k-th lowest of $N-1$ order statistics and $f_k(x)$ denote its corresponding density function. Further, let $f_{k,j}(x,y)$ denote the joint density function of the k-th and j-th lowest of $N-1$ order statistics.

To best represent the CMS rules, it is assumed that all bidders must bid and that when bids at auction are binding, a winning bidder must supply the good even if the price reached at auction is below that bidder's cost.[11] Alternatively, when bids are not binding, bidders observe the auction price and then decide whether to supply the good. In the latter scenario, we use sub-game perfection when examining optimal bids at auction.

The clearing-price auction

Under the assumptions of our model the dynamic clearing price auction proposed in Cramton (2011) is analogous to that studied in Vickrey (1962) where the reimbursement price paid to winners equals the lowest-losing bid. It is well known that in our environment this payment rule

[10] The environment faced by Medicare bidders is much more complex than is our model, likely including common value components as well as multi-unit capacities. We focus on the case where costs are independently distributed and bidders have unit capacities because it admits equilibrium solutions, the results of which are sufficient to conclude that the median price auction is flawed. It is unlikely that the median price auction would somehow become better in a more complex setting.

[11] This also matches the experimental rules set forth in Merlob et al. (2011) and allows for direct comparisons of our results to theirs.

gives bidders a dominant strategy of bidding their cost if bids are binding. The resulting market clearing price equals the $W+1$st lowest cost. Because this price is above each winner's cost, all winners are willing to supply the product. The result is a fully efficient outcome in which the W lowest-cost firms end up supplying the product.

It is well recognized that there are other equilibria in the clearing-price auction where some bidders bid less than their cost. For example, W bidders bidding $\underline{b}$ and all other bidders bidding $\bar{b}$ is a Nash equilibrium. However, equilibria such as this are commonly discounted by using the dominant strategy refinement and noting that bidding below cost is a dominated strategy. However, if bids are not binding, many new lowball bidding equilibria arise that cannot be refined away. Bidding below cost is not dominated when bidders can simply walk away from the contract.

Bidding below cost in the clearing-price auction is dominated when bids are binding because by bidding less than cost, the bidder runs the risk of winning the auction but obtaining a negative payoff. But by bidding costs, a bidder will never receive a negative payoff when winning, and is assured of winning whenever there would be a positive payoff. Once bids are not binding however, bidders need not worry about receiving a negative payoff from a below-cost bid because they are free to walk away from the contract. Because below-cost bids do not earn negative payoffs, bidding below cost is not dominated. It also means that in the extreme, everyone bidding the lowest allowed bid $\underline{b}$ is an un-dominated equilibrium when bids are not binding. Thus, it is not surprising that Merlob et al. (2011) find that bidders in clearing-price auctions do in fact lowball bid in the experimental lab when bids are not binding.

The median-price auction

The median-price auction used by CMS sets the price equal to the median of the W winners' bids. It is assumed that ties (which end up occurring with positive probability in some equilibria) are broken by choosing winners randomly (with equal probability) from those whose bids tied. If bids are not binding, winners may decline to sign a contract with CMS. When this occurs, CMS turns to the lowest-losing bidder and offers that bidder a contract at the original median price. In a strictly increasing equilibrium those bidders to whom offers are subsequently made have higher costs than the firm that first declined the contract and will therefore decline the contract as well.

Full information

When studying auctions it is often useful to examine a full information environment before proceeding to the more realistic environment of incomplete information. Often the full information equilibrium is the limiting case of the incomplete information case and can provide much needed intuition about a problem. For example, this is the case in the clearing-price auction with binding bids where bidding cost is a dominant strategy in both full and incomplete information environments.

We begin by assuming that all bidders know that the costs are $c_1 < c_2 < \cdots < c_N$. When bids are binding in the median-price auction there are a number of equilibria that generate inefficient outcomes. For example, if $W \geq 5$ it is an equilibrium for the $(W+3)/2$ lowest-cost bidders to bid anywhere between $c_{(W+3)/2}$ and $c_{(W+5)/2}$ while all others bid $\bar{b}$. This sets the equilibrium price between $c_{(W+3)/2}$ and $c_{(W+5)/2}$ which results in at least one bidder with a bid of $\bar{b}$ being chosen randomly as a winner, but being reimbursed an amount less than their cost. Further, because some winners are chosen randomly, it is possible that not all of the W lowest-cost bidders are awarded contracts.

When bids are not binding even more equilibria arise in the full information median-price auction. Since bidders are now able to decline contracts, even the extreme case where all bidders bid $\underline{b}$ becomes an equilibrium. More importantly, these equilibria are not dominated. To see that bidding below cost is not dominated (and is in fact payoff equivalent to bidding cost) in the median-price auction when bids are not binding we must consider two cases: $c \geq m$ and $c < m$. When $c \geq m$, bidding c leads to zero profit since the bidder either wins at a price equal to or less than their cost (and declines the contract), or loses. By bidding less than c the bidder increases the chances of winning and may lower the price by changing the median winning bid. But, whenever the bidder wins it is at a price below cost, the bidder declines the supply contract, and continues to earn zero profit. Alternatively, when $c < m$, bidding c and bidding below c both give profit of $m - c$ since both strategies assure the bidder of winning and the median bids are the same under each strategy.

In addition to the inefficient equilibrium mentioned above, there is also an efficient equilibrium in the full information setting where the W low-cost bidders bid c_{W+1}, the bidder with cost c_{W+1} bids $c_{W+1}+\varepsilon$,[12] and all other bidders with cost greater than c_{W+1} bid c_{W+2} or greater. Interestingly, this is an equilibrium whether bids are binding or not. We will see in the next section that this efficient equilibrium is an artifact of the full information assumption and does not carry over into the incomplete information case.

In addition to the incomplete information experiments mentioned above, Merlob et al. (2011) also ran the auctions in a full information environment. Despite the existence of an efficient equilibrium in this setting, the full information, median-price auction was the *worst* performing of all formats tested in terms of efficiency. We conjecture two causes for this poor experimental performance. First, it appears that experimental bidders adopted strategies from different equilibria, some of which consist of lowball bidding. Second, the theoretically efficient equilibrium has bidders with costs above the market clearing price simply bid above the market clearing price and accept losing. It could be that the experimental participants were not willing to bid above the market clearing price simply to support the equilibrium when doing so would result in a zero payoff, particularly since lowball bidding guaranteed them the same payoff while also giving them the option of signing the contract if the price was favorable.

Incomplete information with binding bids

We begin by examining the bidder's problem when there is no ceiling placed on bids to show that the bid ceiling will almost surely bind. In doing so we assume that each of bidder i's opponents is using the strictly increasing bid function $\beta(c)\geq c$ with inverse $\phi(\beta)$. Bidder i's problem (suppressing the bidder subscript i) is to choose the bid, b, in order to maximize

$$\begin{aligned}&\int_{\phi(b)}^{H}(\beta(x)-c)\,f_{(W-1)/2}(x)dx\\&+(b-c)\left[F_{(W-1)/2}(\phi(b))-F_{(W+1)/2}(\phi(b))\right]\\&+\int_{L}^{\phi(b)}\int_{\phi(b)}^{H}(\beta(x)-c)\,f_{(W+1)/2,W}(x,y)dydx\end{aligned}\tag{1}$$

where the first term represents the case in which i's bid wins and is below the price-setting bid, the second term is when i's bid wins and sets the price, and the third term is when i's bid wins and is above the price-setting bid.

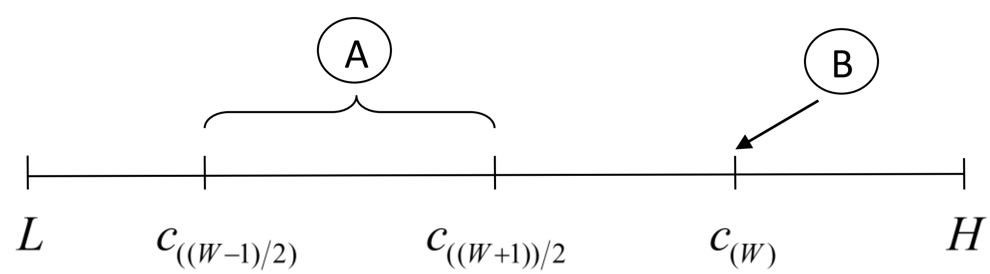
[12] ε represents the fact that the bidder with cost c_{W+1} is aggressively mixing just above c_{W+1} such that none of the bidders with lower cost want to raise their bids. See Hirshleifer and Riley (1992, Chapter 10) for a detailed discussion of the role of mixing in full information auctions.

Taking the derivative of Equation (1) with respect to b, imposing symmetry, and rearranging gives the following equilibrium condition:

$$\beta'(c)\left[F_{(W-1)/2}(c)-F_{(W+1)/2}(c)\right]=\int_L^c(\beta(x)-c)\,f_{(W+1)/2,W}(x,c)dx. \tag{2}$$

Figure 1 presents the intuition for Equation (2). Region A denotes the case where bidder i's bid lies between $c_{((W-1)/2)}$ and $c_{((W+1)/2)}$ and thus sets the price. The LHS of Equation (2) is simply the probability that bidder i's bid falls in region A times the incremental change in the price brought about by a change in his bid. The RHS of Equation (2) represents the instantaneous probability that lowering bidder i's bid makes him a winner (this happens if he is at point B where $c=c_{(W)}$) times the expected payoff i receives from becoming a winner (the value of $(\beta(x)-c)$, integrated over all possible values of the equilibrium price-setting bidder's cost, $c_{((W+1)/2)}$). This equilibrium is dictated by events where either the bidder sets the price, or the price-setting bid is below his own and events where the price is set by a bid above his own do not influence the equilibrium.

Figure 1: Intuition for Equation (2)



Unfortunately, there is no known closed-form solution to the integro-differential equation given by Equation (2). We are however able to obtain solutions in power series form once a cost distribution is specified. With that in mind, we will show that a unique monotone increasing, bounded equilibrium bid function exists in the case when $W=3$ and costs are uniformly distributed. But, we also provide evidence that no such solution exists when $W>3$ and costs are uniformly distributed. Thus indicating that there are relevant settings in which a bounded equilibrium bid function does not exist in the median-price auction.

Despite the lack of a general solution to Equation (2), we are able to obtain general necessary conditions that must be satisfied by any equilibrium solution to that equation. These conditions are given below in Theorem 1. But first, to provide the setting for Theorem 1, it is convenient to define $B=(W-1)/2$ and $A=N-1-W$ whenever bidder i submits a winning bid that is above the price-setting bid. B is simply the number of bids submitted by bidder i's $N-1$ opponents that are below the price-setting bid and A is the number of opponents' bids that are above the lowest-losing bid. The remaining $B-1$ opponents (along with bidder i) submit winning bids that are above the price-setting bid. For example, if bidder i submits a winning bid that is above the price-setting bid when $W=7$ and $N=12$ as in Merlob et al. (2011), then of bidder i's $N-1=11$ opponents, $B=3$ will have bid below the price-setting bid, $A=4$ will have bid above the lowest-losing bid, and $B-1=2$ will have bid between the price-setting bid and the lowest-losing bid.

An essential ingredient of our analysis is the operator $D_c=\frac{1}{f(c)}\frac{d}{dc}$ (differentiation followed by division by $f(c)$). For any $k\geq 1$, the k-fold iterate of D_c will be denoted by D_c^k. Namely, D_c^k allows us to define $\gamma_B(c)=\frac{B!}{(2B)!}\frac{D_c^B\left(cF(c)^{2B}\right)}{F(c)^B}$ which plays a crucial role in our conclusions regarding non-existence. By our assumption that $f(c)$ is positive and has derivatives of all orders throughout $[L,H]$, γ_B is also defined throughout that interval. In addition, it is shown in the Appendix that γ_B can be continuously extended to the entire interval $[L,H]$ and that $\gamma_B(L)=L$.

Theorem 1. *If* β *is a solution of Equation* (2) *on* (L,H) *such that* $\beta'(c)>0$ *for all* $c\in(L,H)$ *and such that* β *is continuous (and hence bounded) on* $[L,H]$*, then*

i) $\beta^{(n)}(L)=0$ *for all* $n=1,2,...,B$.

ii) $\beta(H)=\gamma_B(H)$

iii) $\beta(c)>c$ *for all* $c\in[L,H]$ *and* $L<\beta(L)<L+\frac{W}{f_{\min}N}$.

The proof of Theorem 1 is given in the Appendix. However, we now make some useful observations about its implications. First, we note that repeated application of the operator D_c to Equation (2) yields a linear differential equation of order $B+1$ in β for which a unique solution can be determined if values of $\beta(L)$ and $\beta^{(n)}(L)$ for $n=1,2,...,B$ are specified. Therefore, by establishing that Equation (2) pins down the first B derivatives of β at L, Part i of Theorem 1 guarantees that Equation (2) can have at most one solution for any fixed initial value, $\beta(L)$.

The implication of Part iii of Theorem 1 is intuitively appealing. A small ratio of winners to bidders (W/N), which indicates a high level of competition, forces the lowest-cost bidders to bid aggressively (just above cost) whereas low-cost bidders can bid fairly high when the ratio is close to one. Part ii of the theorem provides the less intuitive conclusion that the equilibrium bid of the highest-cost bidder depends only on the number of winners and *not* the total number of bidders. This is because the function γ_B is defined in terms of B (and hence W) but does not depend on N. Parts i and ii together show that at most one bounded solution exists and that if it does, then Part ii gives the exact value to which the solution converges as $c \to H$.

As stated above, solutions to Equation (2) can be expressed as power series, but doing so requires specifying the distribution of costs. The two examples that follow use the uniform distribution of costs where $F(c)=(c-L)/(H-L)$ and $f(c)=f_{\min}=1/(H-L)$. In this setting, Parts ii and iii of Theorem 1 reduce to $\beta(H)=H+\frac{W-1}{W+1}(H-L)$ and $L<\beta(L)<L+\frac{W}{N}(H-L)$.

Example 1: The case of *W* = 3 winners; *U*[0,1] cost distribution

Although it is unlikely that a CMS auction would have only three winners, our first example considers just such a case because it admits a complete mathematical analysis of the solutions to Equation (2) and sets a comparative foundation for our second example where $W=7$.

Assuming that costs are uniformly distributed on the $[0,1]$ interval, Equation (2) becomes

$$c(1-c)^2\beta'(c)=(N-2)(N-3)\int_0^c x(\beta(x)-c)dx \tag{3}$$

and by Theorem 1 we know that any bounded, monotone increasing solution must satisfy $\beta'(0)=0,\quad \beta(0)<3/N$, and $\beta(1)=1.5$.

It is useful to note that any solution of Equation (3) is also a solution of the second order differential equation

$$c(1-c)^2\beta''(c)+(1-c)(1-3c)\beta'(c)-(N-2)(N-3)c\beta(c)=-1.5(N-2)(N-3)c^2$$

obtained by differentiating Equation (3) with respect to c. Because Part i of Theorem 1 specifies that that $\beta'(0)=0$, this second-order differential equation has a unique solution (on some open interval containing $c=0$) for any given initial value $\beta(0)=b_0$. Furthermore, each of these solutions can be expressed as a power series $\beta(c)\equiv\beta(c,b_0)=b_0+\sum_{n=1}^{\infty}b_nc^n$ where the sequence of coefficients b_n are defined by $b_1=0$, $b_2=b_0/2$, and the three term recurrence relation

$$b_n=\frac{2n(n-1)b_{n-1}-(n^2-2n-2)b_{n-2}}{n^2},n\geq 3.$$

It can be shown using the Ratio Test that for any choice of b_0, the above power series has radius of convergence equal to 1 and that $\beta(c,b_0)$ is a solution of Equation (3) on the interval $(0,1)$. It can also be shown that there is a unique $b_0\equiv b*$ such that $c=1$ is also included in the interval of convergence of $\beta(c,b*)$ and it then follows from Abel's Theorem that $\beta'(c,b*)$ is continuous throughout the interval $[0,1]$.[13] In addition, we discover that, for $b_0=b*$, we have $b_n>0$ for all $n\geq 2$ and this allows us to conclude that $\beta'(c,b*)>0$ and $\beta''(c,b*)>0$ for all $c\in(0,1)$. Further investigation of the power series reveals that the solution becomes infinitely steep as $c\to 1^-$.

We are now prepared to discussed why $W=3$ is more amenable to complete mathematical analysis than are cases where $W>3$. In particular, when $W=3$, $\beta(c,b*)$ can be expressed in the form $\beta(c,b*)=\frac{p(c)+H(r,s,1;c)}{(1-c)^2}$ where $H(r,s,1;c)$ is Gauss' hyper-geometric function with numerator parameters r and s that depend on W and N (see Brand, 1966, pp. 439-440 for a thorough discussion) whereas this is impossible for $W>3$. Well-known properties of H (in particular Gauss' Theorem) can then be used in the $W=3$ case to determine the exact value of $\beta(0,b*)=b*$ for any N. For example, when $N=4$, Gauss' Theorem gives

[13] See Ahlfors (1979), page 41 for details on Abel's Theorem.

$$b^* = 8 - \frac{\Gamma\left(2+\sqrt{3}\right)\Gamma\left(2-\sqrt{3}\right)}{\Gamma(1)\Gamma(3)} \approx 0.704.$$

Figure 2: Solutions to Equation (3) when W=3, N=4

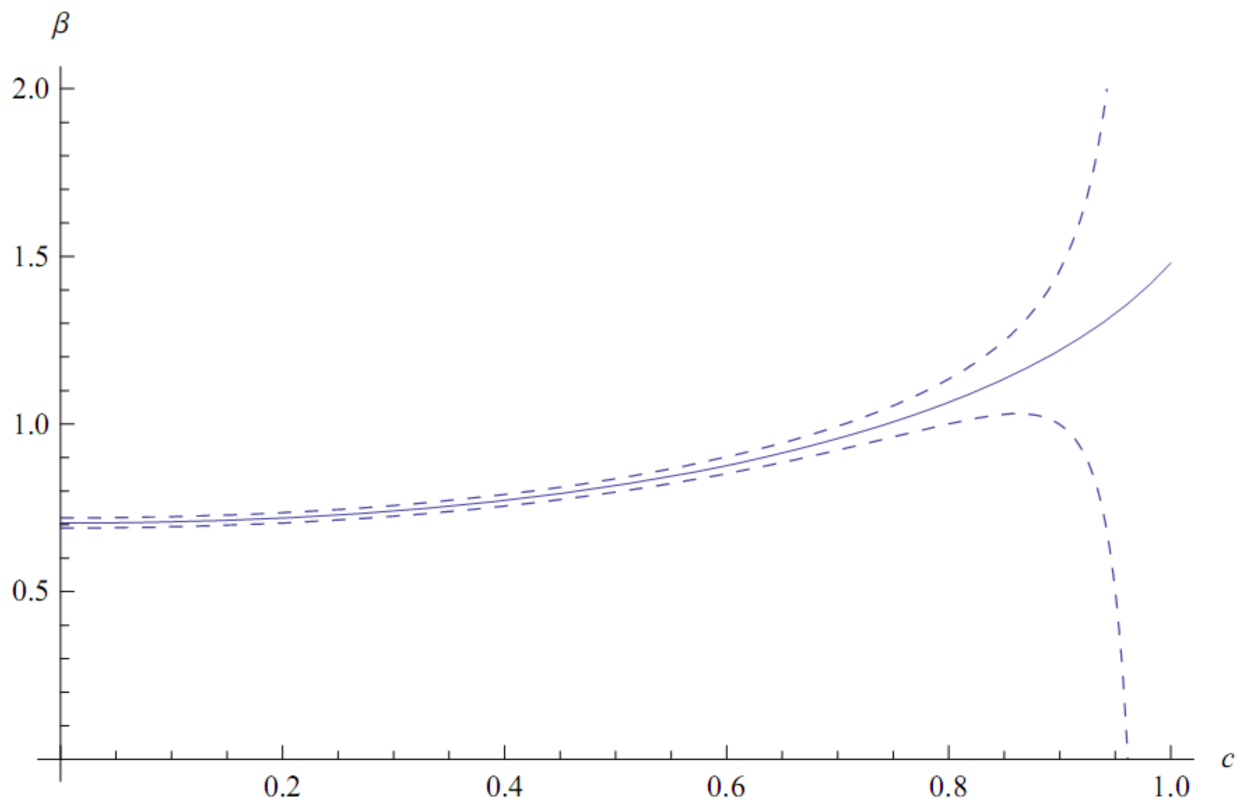


Figure 2 gives a feeling for the behavior of the power series solutions for the specific case where $N = 4$ (solutions for $N > 4$ are of the same basic form). The lower dashed curve begins at $b_0 = 0.69$ and diverges to negative infinity which is representative of all solutions emanating from initial values $\beta(0) = b_0 < b^*$. Likewise, the upper dashed curve begins at $b_0 = 0.72$ and diverges to positive infinity which is representative of all solutions for which $\beta(0) = b_0 > b^*$. Only for $\beta(0) = b^* \approx 0.704$ does the solution converge on the entire interval $[0,1]$ and by Part i of Theorem 1 it converges to $\frac{2W}{W+1} = 1.5$. That solution is represented by the solid curve in Figure 2.[14]

[14] Since the solid bid function equilibrium is monotone increasing and bounded, the median-price auction is "revenue equivalent" under that equilibrium to the clearing-price auction as both auctions generate an expected reimbursement price of 0.8 in this example. However, we point out that the median-price auction may result in bidders supplying products at prices below cost which the clearing-price auction never does.

The practical implication of these results is that equilibrium either results in bids that approach infinity or a convergent bid function where the highest bids are almost one-and-one-half times cost. Clearly in either case the CMS bid ceiling will bind and outcomes will be inefficient. Unfortunately as the next example shows, this sub-optimal result is probably a best-case scenario since it appears that when $W > 3$, a bounded, monotone increasing solution does not even exist.

Example 2: The case with *W* = 7 winners, *U[100,1000]* cost distribution

Our second example examines the case of seven winners. Here we assume that costs are uniformly distributed on $[100,1000]$ and set the number of bidders to $N = 12$. We choose this scenario because it corresponds to the recent experimental work of Merlob et al. (2011) and allows us to shed light on their findings.

In general for the case of seven winners, Equation (2) becomes

$$(c-L)^3(H-c)^4\beta'(c) = K\int_L^c (x-L)^3(c-x)^2(\beta(x)-c)dx. \tag{4}$$

where $K = (N-7)(N-6)(N-5)(N-4)$. For the case where $N = 12$ and $c \sim U[100,1000]$, Theorem 1 indicates that

$$\beta'(100) = \beta''(100) = \beta'''(100) = 0,$$
$$\beta(100) < 100 + \frac{7}{12}(900) = 625,$$
$$\beta(1000) = 1000 + \frac{6}{8}(900) = 1,675.$$

Similar to the $W = 3$ case, integro-differential Equation (4) can be converted into a linear differential equation (of fourth order in this case). Unfortunately, Equation (4) cannot be reformulated as a hyper-geometric equation as in the $W = 3$ case (since this is only possible with linear differential equations of second order) and little is known about analytic solutions to this form of equation other than the fact that it can be expressed as a power series centered at $L = 100$. The solution in this example being $\beta(c,b_0) = \sum_{n=0}^{\infty} b_n(c-100)^n$, with coefficients defined by $b_1 = b_2 = b_3 = 0$, $b_4 = \frac{K(b_0-100)}{480(H-L)^4}$, $b_5 = \frac{K(b_0-L)}{150(H-L)^5} - \frac{K}{600(H-L)^4}$, and the five term recurrence relation

$$b_n = \frac{\left(\frac{K}{n(n+1)(n+2)} - (n-4)\right) b_{n-4} + 4(n-3)(H-L) b_{n-3} - 6(n-2)(H-L)^2 b_{n-2} + 4(n-1)(H-L)^3 b_{n-1}}{n(H-L)^4}$$

for $n \geq 6$.

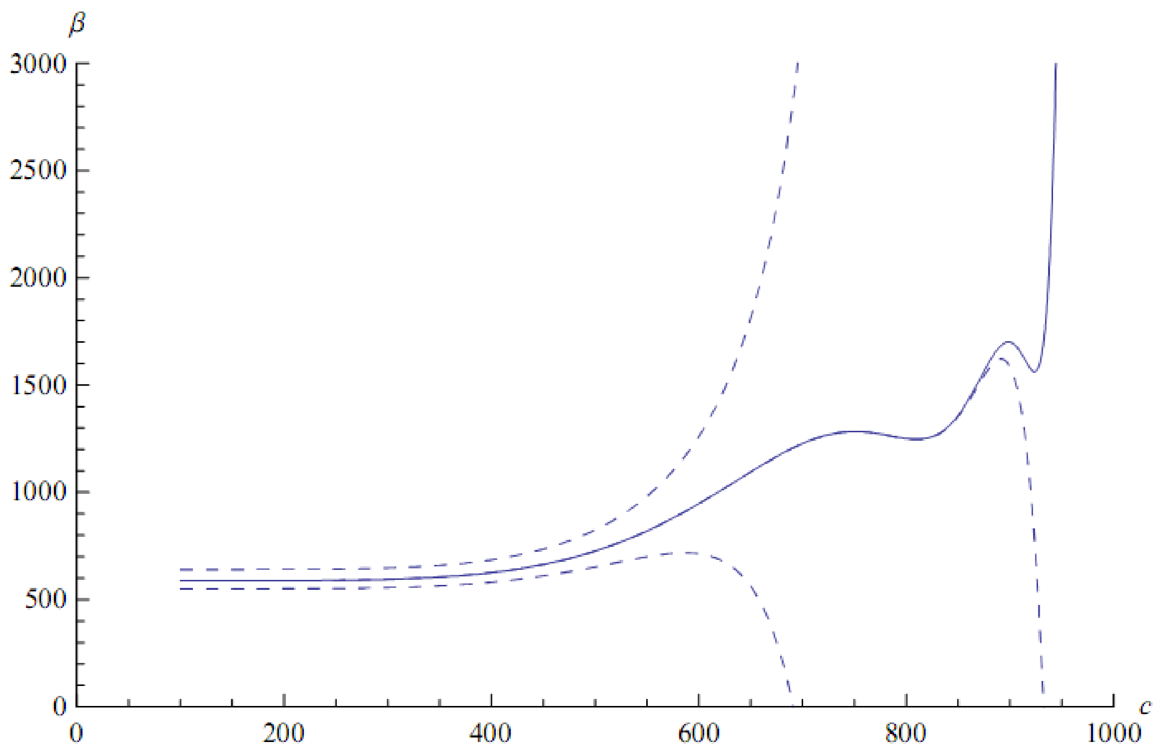
Once again, Part i of Theorem 1, tells us that Equation (4) pins down the first three derivatives and therefore the solutions to the equation are unique for any fixed initial condition $\beta(100)$. Figure 3 graphs the power series solutions for four different choices of b_0 (0.55, 0.588353, 0.588353, and 0.64). It shows that the qualitative nature of the family of solutions of Equation (4) for $W = 7$, $N = 12$ is similar to that in the example where $W = 3$ in that there appears to be a critical value, $b*$, that separates solutions into classes that diverge to positive infinity when $\beta(L) > b*$and negative infinity when $\beta(L) < b*$. For b_0 far enough away from $b*$ (represented by the upper and lower dashed curves), solutions quickly diverge to positive and negative infinity just as in Figure 2. However, for $\beta(L) \neq b*$, but close to $b*$, solutions do not diverge simply to positive or negative infinity like in the $W = 3$ case. Rather, they diverge to $\pm\infty$ non-monotonically.

The non-monotonic behavior of the middle curves in Figure 3 indicate that no bounded equilibrium bid function exists in this example. To see this notice that any solutions starting above or below the middle two curves in Figure 3 cannot approach a finite value. It is only a solution that starts between the two middle curves that can converge, and Part ii of Theorem 1 tells us that solution, with $\beta(L) = b*$, must converge to $\gamma_3(1000) = 1,675$. However, by uniqueness, a curve lying between the middle two curves must approach $1,675$ non-monotonically and we therefore conclude that no bounded equilibrium exists.

This section has supplied convincing mathematical evidence that the median-price auction with binding bids does not admit a bounded, monotone increasing equilibrium under realistic parameter values. We have performed similar analysis for the $W = 5$ and $W = 9$ cases and found similar non-monotonic behavior for values near the critical $b*$ in those cases as well. We conjecture that this non-monotonic behavior is due to the term $(c-x)^{B-1}$ that appears in the integro-differential equation that governs the dynamics. If correct, this would explain the existence of a bounded equilibrium when $W = 3$ as the $(c-x)^{B-1}$ term vanishes since $B - 1 = 0$.

Although not presented here, we have also numerically confirmed that no equilibrium exists in this example where high cost bidders pool bids at $\bar{b}$ and all other bidders use a strictly increasing bid function. It follows that in our examples, the only equilibrium in the binding bids median-price auction with a bid ceiling involves mixed strategies and efficiency is destroyed. This matches well with Merlob et al. (2011) who find that many bids bump up against the bid ceiling while bids below the bid ceiling are highly non-monotonic.

Figure 3: Solutions to Equation (1), W=7, N=12



Incomplete information with non-binding bids

In this section we show that a multitude of equilibria emerge in the median-price auction when bids are not binding. However, as we showed in the section on full information, bidding below cost is not dominated when bids are not binding and these equilibria cannot be refined away. The result is that, absent explicit coordination, individual bidders may adopt strategies from any of these equilibria and the result of the auction is therefore highly unpredictable and likely inefficient.

We construct equilibria by considering situations where bidder i's opponents are using the strictly increasing bid function $\beta(c)$ (with inverse $\phi(b)$) if $c < c^*$ and are bidding $\beta(c^*)$ if $c \geq c^*$

where c^* is uniquely and implicitly defined by $\beta(c^*) = c^*$ for $c^* \in [L, H]$.[15] In essence, this means that bidders with costs below c^* are bidding according to a strictly increasing bid function that eventually crosses the 45^o line where $\beta(c^*) = c^*$ and that bidders with costs higher than c^* all bid the same amount, $\beta(c^*)$.

If bidders are following the strategy described above, then the expected price will always be less than or equal to $\beta(c^*) = c^*$ and bidders with $c \geq c^*$ earn zero expected payoff since they always decline the contract to supply. These bidders cannot profitably deviate by bidding higher than $\beta(c^*)$ since they would win with probability zero and thus still earn zero payoff. Similarly, they cannot profitably deviate by bidding less than $\beta(c^*)$ since the reimbursement price would be less than their cost if they won and hence they would always decline the contract and continue to earn zero payoff. Therefore, bidding $\beta(c^*)$ is equilibrium behavior for bidders with $c \geq c^*$.

For bidders with $c < c^*$, equilibrium bids are derived by examining the following maximization problem for $b < \beta(c^*)$:

$$\begin{aligned}
&\int_{\phi(b)}^{H} (\beta(x) - c) f_{(W-1)/2}(x) dx \\
&+(b - c)\left[F_{(W-1)/2}(\phi(b)) - F_{(W+1)/2}(\phi(b))\right] \\
&+\int_{\max\{L,\phi(c)\}}^{\phi(b)} \int_{\phi(b)}^{H} (\beta(x) - c) f_{(W+1)/2,W}(x, y) dy dx.
\end{aligned} \tag{5}$$

This maximization problem is similar to the binding bids case except that in the last term x is only integrated over the interval $[\max[L, \phi(c)], \phi(b)]$ rather than $[L, \phi(b)]$. The change in the lower limit of integration is a consequence of sub-game perfection which specifies that a bidder will only accept the contract to supply if the auction price ends up being above their cost. Since the last term in Equation (5) is conditioned on the bidder's bid being higher than the price-setting median bid, $\beta(x)$, the maximizer will only accept the contract if that price is greater than their cost, Γ. Or in other words, they only accept the contract if $\beta(x) \geq c \Rightarrow x \geq \phi(c)$.

Imposing symmetry, the FOC condition can be written as

$$\beta'(c)\left[F_{(W-1)/2}(c) - F_{(W+1)/2}(c)\right] = \int_{\max\{L,\phi(c)\}}^{c} (\beta(x) - c) f_{(W+1)/2,W}(x, c) dx. \tag{6}$$

[15] There may be more equilibria than those derived here. However, identifying them is unnecessary as the multiplicity that we identify is sufficient to conclude that the median-price auction performs poorly.

For bidders with cost in the interval $[L,\beta(L)]$, Equation (6) is

$$\beta'(c)\left[F_{(W-1)/2}(c)-F_{(W+1)/2}(c)\right]=\int_L^c(\beta(x)-c)\,f_{(W+1)/2,W}(x,c)dx. \tag{7}$$

which is exactly the same as Equation (2) that was derived above for the case of binding bids. This means that Part i of Theorem 1 applies here as well and the non-binding bids equilibrium bid function must begin with slope of zero and that the resulting solution will be unique for a given choice of $\beta(L)$.

Equilibrium bids for players with cost in the interval $[L,\beta(L)]$ are easily obtained using the appropriate power series solution (similar to those given in Examples 1 and 2). Equilibrium bids by players with costs in the interval $[\beta(L),H]$ are determined by

$$\beta'(c)\left[F_{(W-1)/2}(c)-F_{(W+1)/2}(c)\right]=\int_{\phi(c)}^c(\beta(x)-c)\,f_{(W+1)/2,W}(x,c)dx. \tag{8}$$

However, Equation (8) cannot be solved analytically since it requires inverting the power series solution for bids on the $[L,\beta(L)]$ interval to obtain $\phi(c)$. Fortunately it is straightforward to numerically solve Equation (8) using a forward Euler method by numerically inverting β with Mathematica.[16]

Using the forward Euler method on Equation (8) proceeds as follows. First we obtain the power series solution to Equation (7) which gives all bids on the interval $[L,\beta(L)]$. Then, beginning with $c=\beta(L)$, we calculate $\beta(c+\delta)$ (where δ is the numerical step size) by computing $\beta(\beta(L))$ using the power series and then adding on the incremental change required by Equation (8). This incremental change is easily obtained by rearranging Equation (8) as

$$\beta'(c)=\frac{\int_{\phi(c)}^c(\beta(x)-c)\,f_{(W+1)/2,W}(x,c)dx.}{\left[F_{(W-1)/2}(c)-F_{(W+1)/2}(c)\right]} \tag{9}$$

and multiplying $\beta'(c)$ by δ. Or, $\beta(c+\delta)=\beta(c)+\delta\cdot\beta'(c)$.

We applied this methodology in the setting studied by Merlob et al. (2011) where $c\sim U[100,1000]$ and $N=16$ (the results are similar when using their assumption that $N=12$).

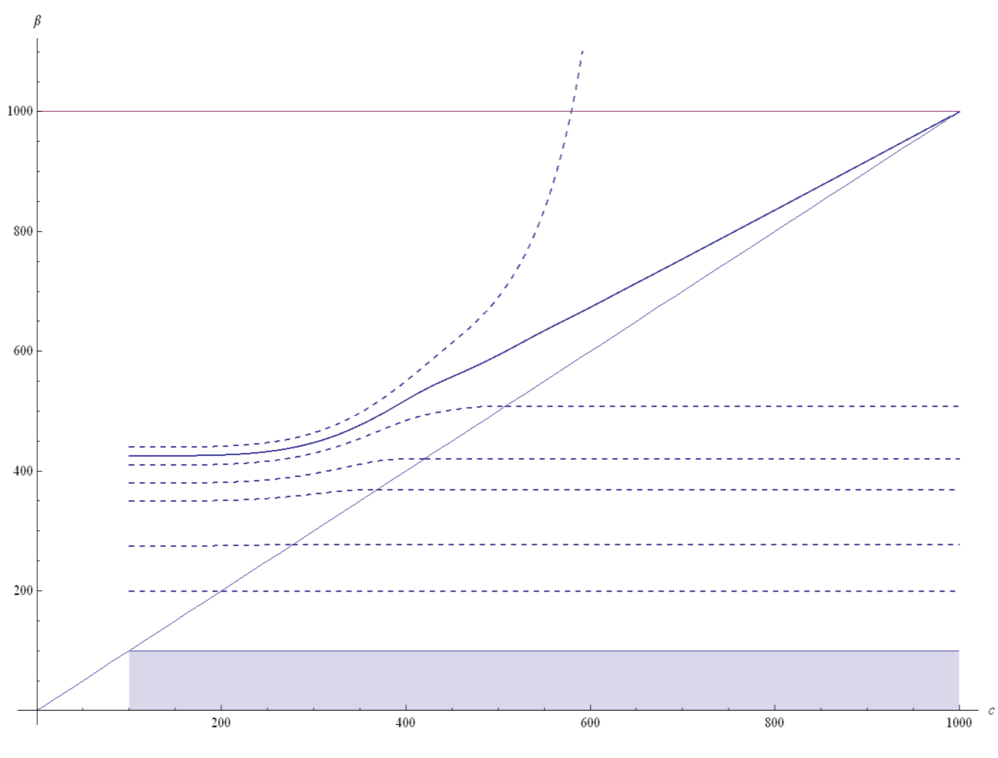
[16] Marshall et al. (1994) provide an excellent discussion of forward and backward Euler methods as they apply to auction problems.

Figure 4 displays eight representative solutions to Equation (6) based on the initial values of $\beta(L)$ found in the first column of Table 1.

Table 1: Non-Binding Bids - Initial and Terminal Bids

$a_0 = \beta(L)$	$c^* = \beta(c^*)$
440.0	NA
425.56299908203551	1000.0
410.0	508.689782000833
380.0	420.737028973323
350.0	368.775566695461
275.0	277.219780623315
200.0	200.103249544203
100.0	100.0

Figure 4: Non-Binding Bids Equilibrium Bid Functions



As in the binding bids model, there is a critical initial value $\beta(100) = b^* \approx 425.56299908203551$ in this case. The increasing solid curve in Figure 4 emanating from that initial value represents the only bounded equilibrium bid function that does not consist of any below cost bids. The highest dashed curve is representative of all solutions to Equation (6) that start at some $\beta(100) > b^*$ as each is monotone increasing and diverges to positive infinity. The other dashed curves show equilibria with initial conditions where $\beta(100) < b^*$. Each of these dashed curves is strictly increasing until it hits the 45^o line where $\beta(c) = c$ and is flat from there onward. Note that the slope of these dashed curves is zero at c^* as is required by Equation (9).

The second column of Table 1 lists the different values of c^* that the various curves obtain which provides insight into the relative slopes of the different equilibrium bid functions. When $\beta(100)$ is close to 100, even the increasing portions of these equilibrium bid functions are very flat. For instance, when the bidder with $c = 100$ bids $\beta(100) = 200$, a bidder with $c = 200.103249544203$ bids only 0.103249544203 more than the $c = 100$ bidder. But, when $\beta(100) = 410$, the equilibrium function becomes much steeper with $\beta(c^*) = 508.689782000833$, nearly 100 units higher than $\beta(100)$.

The final equilibrium bid function shown in Figure 4 is the straight line at $\beta(c) = 100$. This is one of many "lowball" bid equilibria that exist. The idea is that if everyone else is bidding 100, then a bidder cannot win by bidding more than 100 and thus bidding 100 and declining the contract is an equilibrium strategy for that bidder. In fact, any situation where all bids are in the shaded region bounded by $\beta = 0$ and $\beta = 100$ constitutes an equilibrium for the same reason. There are many other equilibria as well, such as any situation where at least five bidders bid less than 100 (no matter what the others bid) and, as discussed above, strategies in all of these equilibria are un-dominated. The only thing that mitigates how low bids can fall is the bid floor $\underline{b}$, which Merlob et al. (2011) show was often binding at $\underline{b} = 50$.

It is worth noting that even if the only non-binding bids equilibrium bid function were the solid, increasing curve in Figure 4, the median-price auction would still be ex ante inefficient. Although the W lowest-cost bidders would be selected as winners (since the equilibrium bid function is monotone increasing), the median winning bid sets a price such that some winning bidders decline the supply contract with positive probability. The existence of many other equilibria only compounds the case against the median-price auction with non-binding bids. Coupled with our previous non-existence results from the median-price binding bids model and the overwhelmingly consistent evidence from Merlob et al. (2011), the conclusion is clear: the CMS auction has serious flaws.

Conclusions

Our analysis generates two basic conclusions. First, when bids are non-binding, many new equilibria arise in both the median-price and clearing-price auctions. Many of these equilibria involve lowball bids that set prices too low and result in supply shortages. Second, symmetric increasing strategies require bids that are greater than the bid ceiling in the median-price auction with binding bids. This leads high cost bidders to pool their bids at the bid ceiling and low-cost bidders to randomize their bids below the bid ceiling, virtually guaranteeing that not all of the lowest-cost providers will win.

All of our results are supported by recent experimental findings at the California Institute of Technology (Merlob et al. 2011). The experiments confirm that non-binding bids leads to lowball bids, particularly by high cost bidders. The result is a dramatic loss of efficiency coming from two sources. First, many low-cost bidders do not win the auction—an allocation problem. Second, because the lowball bids set the prices too low, the required number of suppliers needed to meet demand is not reached. Most bidders decline the contracts and demand goes unmet as a result of the supply shortage. Merlob et al. (2011) also support established findings highlighting the efficiency of the clearing-price auction when bids are binding. Cramton et al. (2010) show that a modern clock implementation of the clearing-price auction further strengthens efficiency and the tendency to bid truthfully.

The evidence is overwhelming that a fundamental change in auction procedure is necessary to avoid catastrophic failure of the Medicare auctions. Rather than simply applying band-aids to the current process as it has done in the past, CMS must make bids binding and abandon the median-pricing rule. Otherwise, supplier bids will have no relation to costs and outcomes will be highly inefficient. Fortunately, the clearing-price auction is a simple, fully efficient alternative that does harness market forces by encouraging bidders to bid their costs. Dynamic clock implementations of the clearing-price auction offer further benefits from price and assignment discovery, especially in the context of auctions for many products. These and other design issues are addressed in Cramton (2011).

Appendix

Preliminaries for the Proof of Theorem 1

A bidder's cost is assumed to be distributed on the interval $[L,H]$ according to the distribution F with density f that is assumed to be positive-valued and to have derivatives of all orders throughout $[L,H]$. The probability that the k-th highest of n costs is less than or equal to c is $F_{(k,n)}(c)$ with corresponding density function

$$f_{(k,n)}(c) = k\binom{n}{k}(1-F(c))^{k-1}F(c)^{n-k}f(c).$$

Notice that unlike in the body of the paper, we use the total number of order statistics, n, in the subscript here. This is because the body of the paper always considered situations where the order statistics were the costs of a bidder's $N-1$ opponents and we dropped that part of the subscript to ease the notation. Here however, we will deal with both that situation and one where the order statistics are the costs of all N bidders and must distinguish between the two.

The operator D_c is defined on the set of all functions g that are continuously differentiable on (L,H) by $D_c g = g'/f$ where $g' = dg/dc$. For functions, g, that are k times continuously differentiable on (L,H), we define $D_c^k g$ to be the k-fold iterate of D_c applied to g. If p and q are functions that are continuous on $[L,H]$ with $p(L)=q(L)$ and $D_c p(c) = D_c q(c)$ for all $c\in(L,H)$, then it follows that $p(c)=q(c)$ for all $c\in[L,H]$. Furthermore, since $D_c p(c)/D_c q(c) = p'(c)/q'(c)$ (assuming that $q'(c)\neq 0$), we can use D_c in place of d/dc when working with L'Hopital's Rule (LR). We will do this frequently in giving the proof of Theorem 1.

For each $B\geq 1$, the function γ_B is defined on $[L,H]$ by

$$\gamma_B(c) = \frac{B!}{(2B)!}\frac{D_c^B\left(cF(c)^{2B}\right)}{F(c)^B}, \quad c\in(L,H)$$

By the following proposition, γ_B can be extended continuously to the interval $[L,H]$ with $\gamma_B(L)=L$.

Proposition 1. *For any* $B\geq 1$ *and* $0\leq n\leq 2B$, *the limits of* $D_c^n\left(cF(c)^{2B}\right)$ *as* $c\to L^+$ *and* $c\to H^-$

both exist and are finite. Furthermore,

$$\lim_{c\to L^+} D_c^n\left(cF(c)^{2B}\right)=0, \quad 0\le n\le 2B-1,$$
$$\lim_{c\to L^+} \frac{D_c^n\left(cF(c)^{2B}\right)}{F(c)^{2B-n}}=\frac{(2B)!}{(2B-n)!}L, \quad 0\le n\le 2B,$$

and

$$\lim_{c\to L^+}\gamma_B(c)=L\,.$$

Proof. The assertions of the proposition are clearly true for $n=0$ so we assume that $1\le n\le 2B$. By expanding $D_c^n\left(cF(c)^{2B}\right)$, we observe that

$$D_c^n\left(cF(c)^{2B}\right)=\frac{(2B)!}{(2B-n)!}cF(c)^{2B-n}+\sum_{j=1}^{n}\binom{n}{j}\frac{(2B)!}{(2B-n+j)}D_c^j(c)F(c)^{2B-n+j}. \tag{10}$$

In addition, by expanding $D_c^j(c)$ we observe that $D_c(c)=1/f(c)$ and $D_c^j(c)=\sum_{k=j+1}^{2j-1}\frac{r_{(j,k)}(c)}{f(c)^k}, \quad j\ge 2$ where each function $r_{(j,k)}(c)$ is a sum of terms whose factors are derivatives of f. Since f is assumed to be positive-valued and to have derivatives of all orders throughout $[L,H]$, then it is clear that the limits of $D_c^j(c)$ as $c\to L^+$ and $c\to H^-$ both exist and are finite. Thus by Equation (10), the limits of $D_c^n\left(cF(c)^{2B}\right)$ as $c\to L^+$ and $c\to H^-$ both exist and are finite. The remaining claims of the proposition then follow immediately from Equation (10). QED

We now present two more propositions that will be used in the proof of Theorem 1.

Proposition 2. *If g is a function such that $D_c^k g(c)\to 0$ as $c\to L^+$ for $1\le k\le n$, then $g^{(k)}(c)\to 0$ as $c\to L^+$ for $1\le k\le n$.*

Proof. The proof will proceed by induction on n. For $n=1$, if $D_c g(c)\to 0$ as $c\to L^+$, then since $g'(c)=f(c)D_c g(c)$ and $f(L)>0$, it follows that $g'(c)\to 0$ as $c\to L^+$.

For any $n\ge 1$, expansion of $D_c^{n+1}g(c)$ yields

$$D_c^{n+1}g(c)=\frac{g^{(n+1)}(c)}{f(c)^{n+1}}+\sum_{j=n+2}^{2n+1}\frac{r_{(n+1,j)}(c)}{f(c)^j}$$

where each $r_{(n+1,j)}(c)$ is a sum of terms whose factors are derivatives of f and derivatives of g of order less than $n+1$. Furthermore, derivatives of g are present in each of these terms.

Now assume the proposition (the induction hypothesis) to hold for n and suppose that $D_c^k g(c) \to 0$ as $c \to L^+$ for $1 \le k \le n+1$. Then by the induction hypothesis we have $g^{(k)}(c) \to 0$ as $c \to L^+$ for $1 \le k \le n$ and hence

$$\lim_{c \to L^+} \sum_{j=n+2}^{2n+1} \frac{r_{(n+1,j)}(c)}{f(c)^j} = 0.$$

Since $D_c^{n+1} g(c) \to 0$ as $c \to L^+$ and $f(L) > 0$, then $g^{(n+1)}(c) \to 0$ as $c \to L^+$ and the induction argument is complete. QED

Proposition 3. *For any* $B \ge 1$ *and any* $c \in [L, H]$,

$$\int_L^c F(u)^B \left(F(c) - F(u)\right)^{B-1} f(u)\,du = \frac{B!(B-1)!}{(2B)!} F(c)^{2B}.$$

Proof. The assertion of the proposition is clearly true when $B = 1$ so we assume $B > 1$. Let $p(c)$ and $q(c)$ be, respectively, the expressions on the left and right hand sides of the identity that is to be proved. Clearly $p(L) = q(L) = 0$. Also,

$$D_c p(c) = \int_L^c F(u)^B (B-1)\left(F(c) - F(u)\right)^{B-2} f(u)du$$

and

$$D_c q(c) = \frac{B!(B-1)!}{(2B-1)!} F(c)^{2B-1}$$

and hence, $D_c p(L) = D_c q(L) = 0$. By continuing to apply D_c we find that $D_c^n p(L) = D_c^n q(L) = 0$ for $0 \le n \le B-1$ and $D_c^B p(c) = D_c^B q(c) = (B-1)! F(c)^B$ for all $c \in [L, H]$. Since $D_c^{B-1} p(L) = D_c^{B-1} q(L)$, then $D_c^{B-1} p(c) = D_c^{B-1} q(c)$ for all $c \in [L, H]$. By continuing this reasoning, we conclude that $p(c) = q(c)$ for all $c \in [L, H]$. QED

Proof of Theorem 1, Part i

By Proposition 3, Equation (2) can be written as

$$p(c)D_c\beta(c)=R(c) \tag{11}$$

where

$$K=\frac{A!}{(N-1-B)!},$$

$$p(c)=KF(c)^B\left(1-F(c)\right)^{B+1}, \text{ and}$$

$$R(c)=\frac{1}{(B-1)!}\int_L^c F(u)^B\left(F(c)-F(u)\right)^{B-1}\beta(u)f(u)\,du-\frac{B!}{(2B)!}cF(c)^{2B}. \tag{12}$$

Direct computation gives

$$\begin{aligned} D_c^nR(c)=&\frac{1}{(B-1-n)!}\int_L^c F(u)^B\left(F(c)-F(u)\right)^{B-n-1}\beta(u)f(u)du \\ &-\frac{B!}{(2B)!}D_c^n\left(cF(c)^{2B}\right),\quad 0\le n\le B-1 \end{aligned}$$

and $D_c^BR(c)=F(c)^B\left(\beta(c)-\gamma_B(c)\right)$.

Since $\lim_{c\to L^+}D_c^n\left(cF(c)^{2B}\right)=0$ for $0\le n\le 2B-1$ by Proposition 1, then $\lim_{c\to L^+}D_c^nR(c)=0$ for $0\le n\le B$ and we can apply L'Hopital's Rule (using the operator D_c in place of d/dc) to obtain

$$\begin{aligned} \lim_{c\to L^+}\frac{(2B)!R(c)}{B!F(c)^{2B}}&=\lim_{c\to L^+}\frac{(2B-1)!D_cR(c)}{B!F(c)^{2B-1}} \\ &=\lim_{c\to L^+}\frac{(2B-2)!D_c^2R(c)}{B!F(c)^{2B-2}} \\ &\vdots \\ &=\lim_{c\to L^+}\frac{(B+1)!D_c^{B-1}R(c)}{B!F(c)^{B+1}} \\ &=\lim_{c\to L^+}\frac{D_c^BR(c)}{F(c)^B} \\ &=\lim_{c\to L^+}\left(\beta(c)-\gamma_B(c)\right) \\ &=\beta(L)-L \end{aligned}$$

by Proposition 1 and the assumption that β is continuous at $c=L$. We conclude that

$$\lim_{c\to L^+}\frac{D_c^nR(c)}{F(c)^{2B-n}}=\frac{B!}{(2B-n)!}(\beta(L)-L)\ \text{ for }\ 0\le n\le B. \tag{13}$$

In addition, expansion of $D_c^n p(c)$ gives

$$D_c^n p(c) = \frac{KB!}{(B-n)!} F(c)^{B-n} + K \sum_{j=1}^{B+1} (-1)^j \binom{B+1}{j} \frac{(B+j)!}{(B+j-n)!} F(c)^{B+j-n}$$

for $0 \le n \le B$ and thus

$$\lim_{c \to L^+} \frac{D_c^n p(c)}{F(c)^{B-n}} = \frac{KB!}{(B-n)!} \text{ for } 0 \le n \le B. \tag{14}$$

We will now show by induction on n that if $B \ge n$ and $1 \le k \le n$, then

$$\lim_{c \to L^+} \frac{D_c^k \beta(c)}{F(c)^{B-k+1}} \text{ exists and is finite} \tag{15}$$

and

$$\beta^{(k)}(L) = \lim_{c \to L^+} \beta^{(k)}(c) = 0. \tag{16}$$

The base case in our inductive proof is $B \ge 1$ and $k = 1$. In this case, we divide both sides of Equation (11) by $F(c)^{2B}$ to obtain

$$\frac{p(c)}{F(c)^B} \frac{D_c \beta(c)}{F(c)^B} = \frac{R(c)}{F(c)^{2B}}.$$

Since

$$\lim_{c \to L^+} \frac{p(c)}{F(c)^B} = K$$

by (14) and

$$\lim_{c \to L^+} \frac{R(c)}{F(c)^{2B}} = \frac{B!}{(2B)!} \left(\beta(L) - L \right)$$

by (13), then

$$\lim_{c \to L^+} \frac{D_c \beta(c)}{F(c)^B} = \frac{B!}{K(2B)!} \left(\beta(L) - L \right)$$

which establishes Assertion (15) in the case $n = 1$. Since the above limit is finite, we also have $D_c \beta(c) \to 0$ as $c \to L^+$ and hence $\beta'(c) \to 0$ as $c \to L^+$ by Proposition 2. In addition, since β is assumed to be continuous at $c = L$, then

$$\lim_{c\to L^+}\frac{\beta(c)-\beta(L)}{c-L}=\lim_{c\to L^+}\frac{\beta'(c)}{1}=0$$

which shows both that $\beta'(L)=0$ and that β' is continuous at $c=L$.

Now assume that the induction hypothesis $B\ge n \text{ and } 1\le k\le n \Rightarrow$ (15) and (16) holds for n and suppose that $B\ge n+1$ and $1\le k\le n+1$. Then both (15) and (16) hold for $1\le k\le n$ by the induction hypothesis. Since

$$p(c)D_c^{n+1}\beta(c)+\sum_{j=1}^{n}\binom{n}{j}D_c^j\, p(c)D_c^{n+1-j}\beta(c)=D_c^n R(c),$$

we have

$$\frac{p(c)}{F(c)^B}\frac{D_c^{n+1}\beta(c)}{F(c)^{B-n}}+\sum_{j=1}^{n}\binom{n}{j}\frac{D_c^j\, p(c)}{F(c)^{B-j}}\frac{D_c^{n+1-j}\beta(c)}{F(c)^{B-(n-j)}}=\frac{D_c^n R(c)}{F(c)^{2B-n}}.$$

Also, since

$$\lim_{c\to L^+}\frac{p(c)}{F(c)^B}=K,$$

$$\lim_{c\to L^+}\frac{D_c^n R(c)}{F(c)^{2B-n}}=\frac{B!}{(2B-n)!}\big(\beta(L)-L\big)$$

by (13),

$$\lim_{c\to L^+}\frac{D_c^j\, p(c)}{F(c)^{B-j}}=\frac{KB!}{(B-j)!},\ 1\le j\le n$$

by (14), and

$$\lim_{c\to L^+}\frac{D_c^{n+1-j}\beta(c)}{F(c)^{B-(n-j)}}\quad \text{exists and is finite for } 1\le j\le n$$

by the induction hypothesis, we obtain

$$\lim_{c\to L^+}\frac{D_c^{n+1}\beta(c)}{F(c)^{B-n}}\quad \text{exists and is finite.}$$

This shows that (15) holds for $n+1$ and also shows that $D_c^{n+1}\beta(c)\to 0$ as $c\to L^+$. We also have that $\beta^{(k)}(L)=\lim_{c\to L^+}\beta^{(k)}(c)=0$ for $1\le k\le n$ by the induction hypothesis. By Proposition 2, we conclude that $\beta^{(k)}(c)\to 0$ as $c\to L^+$ for $1\le k\le n+1$. Finally,

$$\lim_{c \to L^+} \frac{\beta^{(n)}(c) - \beta^{(n)}(L)}{c - L} = \lim_{c \to L^+} \frac{\beta^{(n+1)}(c)}{1} = 0$$

shows both that $\beta^{(n+1)}(L) = 0$ and that $\beta^{(n+1)}$ is continuous at $c = L$. QED

Proof of Theorem 1, Part ii

To prove Part ii of Theorem 1, we first observe that because β is continuous at $c = H$ we have

$$\begin{aligned}
\infty &> \beta(H) \\
&= \lim_{c \to H^-} \beta(c) \\
&= \lim_{c \to H^-} \frac{\left(1 - F(c)\right)\beta(c)}{1 - F(c)} \\
&= \lim_{c \to H^-} \frac{\left(1 - F\left(c\right)\right) D_c\beta(c) - \beta(c)}{-1} \\
&= \lim_{c \to H^-} \left(\beta(c) - \left(1 - F(c)\right) D_c\beta(c)\right)
\end{aligned}$$

if this limit exists. Clearly it cannot be the case that $\lim_{c \to H^-} \left(1 - F(c)\right) D_c\beta(c)$ is ∞ or any finite number other than zero because this would contradict L'Hopital's Rule. Therefore either $\lim_{c \to H^-} \left(1 - F(c)\right) D_c\beta(c) = 0$ or this limit does not exist. To determine which is the case, we use Equation (11) to obtain

$$\lim_{c \to H^-} KF(c)^B \left(1 - F(c)\right) D_c\beta(c) = \lim_{c \to H^-} \frac{R(c)}{\left(1 - F(c)\right)^B}.$$

Since $\lim_{c \to H^-} R(c)$ exists and is finite by Proposition 1 and the assumption that β is continuous throughout $[L, H]$, then the limit on the right of the above equation must be equal to zero (for otherwise it would be ∞ which would contradict what was stated above). This implies that both $R(c) \to 0$ and $\left(1 - F(c)\right) D_c\beta(c) \to 0$ as $c \to H^-$. Hence we can apply L'Hopital's Rule to obtain

$$\begin{aligned}
0 &= \lim_{c \to H^-} KF(c)^B \left(1 - F(c)\right) D_c\beta(c) \\
&= \lim_{c \to H^-} \frac{R(c)}{\left(1 - F(c)\right)^B} \\
&= \lim_{c \to H^-} \frac{D_c R(c)}{-B\left(1 - F(c)\right)^{B-1}}
\end{aligned}$$

if the latter limit exists. However, $\lim_{c \to H^-} D_c R(c)$ exists and is finite (by Proposition 1 and the assumption that β is bounded throughout $[L,H]$) and hence it must be the case that $\lim_{c \to H^-} D_c R(c) = 0$ so as not to contradict L'Hopital's Rule. By continuing along this line of reasoning, we obtain

$$
\begin{aligned}
0 &= \lim_{c \to H^-} KF(c)^B \left(1 - F(c)\right) D_c \beta(c) \\
&= \lim_{c \to H^-} \frac{R(c)}{\left(1 - F(c)\right)^B} \\
&\vdots \\
&= \lim_{c \to H^-} (-1)^B \frac{D_c^B R(c)}{B!} \\
&= \lim_{c \to H^-} \frac{F(u)^B \left(\beta(u) - \gamma_B(u)\right)}{B!} \\
&= \frac{\left(\beta(H) - \gamma_B(H)\right)}{B!}
\end{aligned}
$$

and conclude that $\beta(H) = \gamma_B(H)$. QED

It is also easily seen by direct computation that $\gamma_B(H) = H + \frac{W-1}{W+1}(H-L)$ when F is the uniform distribution.

Proof of Theorem 1, Part iii

If β is a bounded and monotone increasing equilibrium for the median-price auction, then a bidder whose cost is c and who bids $\beta(c)$ has expected payoff

$$
\begin{aligned}
&\pi(c) = P\left(c, \beta(c)\right) = \\
&\int_c^H f_{(N-B,N-1)}(u)\left(\beta(u) - c\right) du \\
&+ \binom{N-1}{B} F(c)^B \left(1 - F(c)\right)^{N-1-B} \left(\beta(c) - c\right) \\
&+ \int_L^c \int_c^H \frac{(N-1)!}{B!(B-1)!A!} F(u)^B \left(F(y) - F(u)\right)^{B-1} \left(1 - F(y)\right)^A \left(\beta(u) - c\right) f(u) f(y) dy du.
\end{aligned}
$$

By differentiating and using the first order condition for equilibrium, Equation (2), we obtain

$$\begin{aligned}\pi'(c) = &-\int_c^H f_{(N-B,N-1)}(u)du \\ &-\binom{N-1}{B}F(c)^B\left(1-F(c)\right)^{N-1-B} \\ &-\int_L^c\int_c^H \frac{(N-1)!}{B!(B-1)!A!}F(u)^B\left(F(y)-F(u)\right)^{B-1}\left(1-F(y)\right)^A f(u)f(y)dydu\end{aligned}$$

which shows that $\pi'(L) = -1$ and $\pi'(H) = 0$. By differentiating again we obtain $\pi''(c) = f_{(N-W,N-1)}(c)$.

These observations yield the following lemma.

Lemma 4. *If β is a bounded and monotone increasing equilibrium for the median price auction and all bidders bid according to β, then the expected profit for a bidder of cost $c \in [L,H]$ is*

$$\pi(c) = \int_c^H (1 - F_{(N-W,N-1)}(u))\,du.$$

Proof. We have shown above that $\pi''(c) = f_{(N-W,N-1)}(c)$ and that $\pi'(L) = -1$. This implies that $\pi'(c) + 1 = \int_L^c f_{(N-W,N-1)}(u)\,du = F_{(N-W,N-1)}(c)$. In addition, since $\pi(H) = 0$ we obtain $(0 - \pi(c)) + (H - c) = \int_c^H F_{(N-W,N-1)}(u)\,du$ or $\pi(c) = \int_c^H (1 - F_{(N-W,N-1)}(u))\,du.$ QED

Corollary 5. *If β is an equilibrium for the median price auction, then the expected profit of the lowest cost bidder satisfies $f_{\min}\pi(L) \le \dfrac{W}{N} \le f_{\max}\pi(L)$. Hence, in the case of the uniform distribution ($f(c) \equiv 1/(H-L)$), we have $\pi(L) = \dfrac{W}{N}(H-L)$.*

Proof. By Lemma 4 we have $\pi(L) = \int_L^H (1 - F_{(N-W,N-1)}(u))du$ and since the integrand is positive we obtain

$$f_{\min}\pi(L) \le \int_L^H (1 - F_{(N-W,N-1)}(u))f(u)du = 1 - \int_L^H F_{(N-W,N-1)}(u)f(u)du.$$

Also, since (in general)

$$F_{(k,n)}(u)\,f(u)=\sum_{j=0}^{k-1}\binom{n}{j}\left(1-F(u)\right)^{j}F(u)^{n-j}\,f(u)=\frac{1}{n+1}\sum_{j=0}^{k-1}f_{(j+1,n+1)}(u),$$

then

$$F_{(N-W,N-1)}(u)\,f(u)=\frac{1}{N}\sum_{j=0}^{N-W-1}f_{(j+1,N)}(u)$$

and we obtain

$$\int_L^H F_{(N-W,N-1)}(u)\,f(u)du=\frac{1}{N}\sum_{j=0}^{N-W-1}F_{(j+1,N)}(H)=\frac{N-W}{N}.$$

Hence, $f_{\min}\pi(L)\leq 1-\frac{N-W}{N}=\frac{W}{N}$. The proof of the second assertion of the corollary is similar. QED

We now give the proof of Part iii of Theorem 1. First, to show that $\beta(c)>c$ for all $c\in(L,H]$ we let $c\in(L,H]$ be arbitrary and refer to Equation (2). Since $\beta'(c)>0$, the integral on the right of Equation (2) is positive and hence there must exist some point $u^*\in[L,c)$ such that $\beta(u^*)-c>0$. Since $c>u^*$ and β is monotone increasing on $[u^*,c]$, we thus have that $\beta(c)>c.$ Since $G(c)=\beta(c)-c>0$ for all $c\in(L,H]$ and $G'(L)=\beta'(L)-1=-1$ by Part i of Theorem 1, then it cannot be the case that $G(L)=\beta(L)-L=0$ because this would contradict the fact that $G(c)>0$ throughout $(L,H]$. Therefore $\beta(L)>L$.

To complete the proof, we use the assumption that β is monotone increasing throughout $[L,H]$ and Corollary 5 to obtain

$$\begin{aligned}\beta(L)-L&=\int_L^H f_{(N-B,N-1)}(u)(\beta(L)-L)du\\&<\int_L^H f_{(N-B,N-1)}(u)(\beta(u)-L)du\\&=\pi(L)\leq\frac{W}{f_{\min}N}\end{aligned}$$

which shows that ε. QED

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